S.-T. Yau College Student Mathematics Contest

Applied Mathematics, Individual, 2014

Find the eigenvalues and eigenvectors of the following $N \times N$ tridiagonal matrix

$$A = \begin{pmatrix} b & c & & & & \\ a & b & c & & & \\ & a & b & c & & \\ & & \ddots & \ddots & \ddots & \\ & & & a & b & c \\ & & & & a & b \end{pmatrix}$$

where a, b and c are 3 constants, and $a\,c > 0$.

Definition:

* A *propre k-edge-coloring* of a graph $G(V, E)$ is a mapping $f\colon\ E \to \{1, 2, 3, \cdots, k\}$ such that $f(e) \neq f(e')$ for any pair of edges e,e' that have a common end vertex.

** Suppose that f is a propre k-edge-coloring of a graph $G(V, E)$. f is called a *$uniform$ propre k-edge-coloring* of $G(V, E)$ if for any $i, j \in \{1, 2, 3, \cdots, k\}$, $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$.

Problem:

Prove that if a graph $G(V, E)$ has a propre k-edge-coloring, then $G(V, E)$ has a uniform propre k-edge-coloring.

Consider the following equation over an one-dimensional (1-D) domain $\Omega = (0, 1)$:

$$\partial_t \phi = -\phi^3 + \phi + \epsilon^2 \phi_{xx}, \quad \text{in } \Omega, \tag{1}$$

$$\phi_x = 0, \quad \text{at } x = 0,\, x = 1, \tag{2}$$

with $\epsilon > 0$ a given constant.

The following semi-implicit, semi-discrete numerical scheme is formulated:

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = -\left(\phi^{n+1}\right)^3 + \phi^n + \epsilon^2 \phi_{xx}^{n+1}, \quad \text{in } \Omega, \tag{3}$$

$$\phi_x^{n+1} = 0, \quad \text{at } x = 0,\, x = 1. \tag{4}$$

in which ϕ^k denotes the numerical solution at t^k, with $t^k = k\Delta t$, Δt being the time step size.

Prove the following energy stability for the numerical solution (3)-(4):

$$E(\phi^{n+1}) \le E(\phi^n), \quad \text{for any } \Delta t > 0, \tag{5}$$

with the energy functional given by

$$E(\phi) = \int_\Omega \left(\frac{1}{4}\phi^4 - \frac{1}{2}\phi^2 + \frac{\epsilon^2}{2}|\phi_x|^2 \right) dx = \frac{1}{4}\|\phi\|_{L^4}^4 - \frac{1}{2}\|\phi\|_{L^2}^2 + \frac{\epsilon^2}{2}\|\phi_x\|_{L^2}^2 . \tag{6}$$

Hint. Take an L^2 inner product with (3) by $\tilde{\mu}^{n+1} = \left(\phi^{n+1}\right)^3 - \phi^n - \epsilon^2 \phi_{xx}^{n+1}$.

Oral exam, applied and computational mathematics, individual, 2014

1 Problem 1. Discrete Optimal Mass Transportation

Suppose $\Omega \subset \mathbb{R}^2$ is a convex planar domain, $P = \{p_1, p_2, \cdots, p_n\}$ are discrete points on $\mathbb{R}^2$, each point has a Dirac measure $\{A_i \delta(p - p_i)\}$, such that

$$Area(\Omega) = \sum_{i=1}^{n} A_i.$$

A mapping $f : \Omega \to P$ is called *measure preserving*, if the area of the pre-image of p_i equals to A_i,

$$Area(f^{-1}(p_i)) = A_i.$$

The transportation cost of a mapping is given by

$$E(f) := \int_\Omega |p - f(p)|^2 dp.$$

Among all the measure preserving mappings, the one which minimizes the transportation cost is called *the optimal mass transportation map*.

We want to show that: if f is the optimal mass transportation map, then there exists a convex function $u : \Omega \to \mathbb{R}$, such that f is the gradient map of u, $f = \nabla u$.

1. Let $H = \{h_1, h_2, \cdots, h_n\}$ be weights. The *power voronoi diagram* induced by (P, H) is a cell decomposition of $\mathbb{R}^2$,

$$\mathbb{R}^2 = \bigcup_{i=1}^{n} W_i,$$

where

$$W_i = \{q \in \mathbb{R}^2 | \; |q - p_i|^2 + h_i \leq |q - p_j|^2 + h_j, \forall 1 \leq j \leq n\}.$$

Define a map: $\varphi : W_i \to p_i$. Show that there is a piecewise linear convex function $u : \Omega \to \mathbb{R}$, such that

$$W_i = \{q \in \mathbb{R}^2 \; |\nabla u(q) = p_i\},$$

namely φ is the gradient map of u.

2. Suppose there is another cell decomposition

$$\mathbb{R}^2 = \bigcup_{i=1}^{n} \overline{W}_i,$$

such that

$$Area(W_i \cap \Omega) = Area(\overline{W}_i \cap \Omega),$$

The map induced by this cell decomposition is $\bar{\varphi} : \overline{W}_i \to p_i$. prove the transportation cost of φ is no greater than that of $\bar{\varphi}$,

$$\int_\Omega |q - \varphi(q)|^2 dq \leq \int_\Omega |q - \bar{\varphi}(q)|^2 dq.$$

Namely, the discrete optimal mass transportation map must be induced by a power voronoi diagram.

3. Suppose given any $A = \{A_1, A_2, \cdots, A_n\}$, such that $A_i > 0$ and $\sum_{i=1}^{n} A_i = Area(\Omega)$, we can always find H, such that the power voronoi diagram induced by H satisfies the condition $Area(W_i) = A_i$, then show that the discrete optimal mass transportation is given by the gradient map of a convex function.

2 Problem 2. Circle Packing

A discrete surface is represented as a simplicial complex, such that each face is a Euclidean triangle, which is also called a triangle mesh. Suppose $M = (V, E, F)$ is a triangle mesh, where V, E, F represents the set of vertices, edges and faces respectively. The Euler number of the mesh is $\chi(M) = |V| + |F| - |E|$. Furthermore, a circle packing defined on the mesh. Each vertex v_i is associated with a circle (v_i, r_i), two circles on an edge are tangent to each other.

Figure 1: A discrete surface is represented as a triangle mesh.

Suppose $v_i \in V$ is an interior vertex on M, $[v_i, v_j, v_k] \in F$ is a face on M. θ_i^{jk} is the corner angle on the face $[v_i, v_j, v_k]$ with apex v_i. The discrete curvature at v_i is defined as

$$K_i = 2\pi - \sum_{[v_i,v_j,v_k]\in F} \theta_i^{jk}$$

The the total curvature satisfies the Gauss-Bonnet theorem $\sum_i K_i = 2\pi\chi(M)$. Let $u_i = \log r_i$, which is called the discrete conformal factor. We want to show the mapping from the discrete conformal factor to the discrete curvature

$$\varphi : (u_1, u_2, \cdots, u_n) \mapsto (K_1, K_2, \cdots, K_n),$$

where $\sum_i u_i = 0$, is deffeomorphic.

1. Derivative Cosine Law

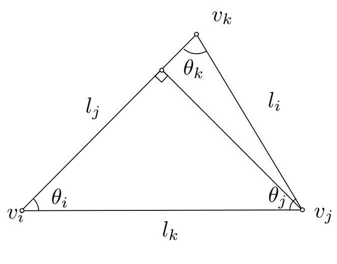

Figure 2: A Euclidean triangle.

Consider one triangle $[v_i, v_j, v_k]$, the corner angles are the functions of edge lengths, $\theta_i(l_i, l_j, l_k)$, prove

$$\frac{\partial\theta_i}{\partial l_i} = \frac{l_i}{2A}, \frac{\partial\theta_i}{\partial l_j} = -\frac{l_i}{2A}\cos\theta_k,$$

where A is the triangle area.

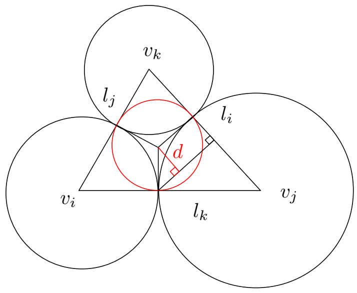

Figure 3: A Euclidean triangle with a circle packing.

2. Circle Packing on one Triangle. Suppose we associate each vertex v_i with a circle $c_i(v_i, r_i)$ centered at v_i with radius r_i. All three circles are tangent to each other, the inner circle has radius r, let $u_i = \log r_i$, prove

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \frac{r}{l_k}$$

and

$$\frac{\partial \theta_i}{\partial u_i} = -\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_i}{\partial u_k}.$$

Prove that the mapping

$$\varphi : \{(u_i, u_j, u_k) | u_i + u_j + u_k = 0\} \rightarrow \{(\theta_i, \theta_j, \theta_k) | \theta_i + \theta_j + \theta_k = \pi\}$$

is a diffeomorphism.

3. Consider the whole triangle mesh, prove the mapping

$$\varphi : (u_1, u_2, \cdots, u_n) \mapsto (K_1, K_2, \cdots, K_n),$$

where $\sum_i u_i = 0$, is deffeomorphic.